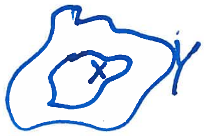

An abbrev: "$x \subseteq y$" stands for $\forall z(z \in x \to z \in y)$

~~Theorem:~~ ~~$(x \subseteq y \wedge y \subseteq x) \to x = y$~~

~~Proof~~: How to handle $\exists$ and $\forall$:

- remove $\exists x\, \varphi(x)$ by writing $\varphi(x_0)$ — if x_i already used, use x_{i+1} instead
- remove $\forall x \varphi(x)$ by writing $\varphi(z)$ for any z you want
- $\varphi(w)$ can be replaced by $\exists x \varphi(x)$
- $\varphi(w)$ can be replaced by $\forall x \varphi(x)$ only when w didn't come from removing $\exists$ earlier

Theorem: The empty set exists.

Proof: Take $\varphi(x)$ to be ~~mean~~ ~~stand for~~ $\neg(x=x)$.

By Axiom 2,

$$\exists z \forall x (x \in z \leftrightarrow \neg(x=x))$$

call it z_0 — pick one (arbitrary) call it x

$$x \in z_0 \leftrightarrow \neg(x=x)$$

$\leftrightarrow$-def $\qquad$ $\leftrightarrow$-def

$$x \in z_0 \to \neg(x=x) \qquad \neg(x=x) \to x \in z_0$$

$$\Downarrow$$

$$\neg\neg(x=x) \to \neg(x \in z_0)$$

$$\Downarrow$$

$$x=x \to \neg(x \in z_0)$$

taut. true!

$$\therefore \neg(x \in z_0) \to \forall x(\neg(x \in z_0))$$

$$\to \exists z \forall x(\neg(x \in z))$$

we call this set z "the empty set"

we give it a special symbol — $\emptyset$

Empty set theorem: $\forall x\ \neg(x \in \emptyset)$

<u>Thm</u>: $x \subseteq y \land y \subseteq x \xrightarrow{\text{then}} x = y$

<u>Pf</u>: Assume $x \subseteq y \land y \subseteq x$, ~~i.e.~~ from which we may conclude

(i) $x \subseteq y$, and

(ii) $y \subseteq x$.

By def of $\subseteq$, we get

(i) $\forall z(z \in x \to z \in y)$, and

(ii) $\forall z(z \in y \to z \in x)$

Thus, removing $\forall$,

(i) $z \in x \to z \in y$

(ii) $z \in y \to z \in x$

By $\leftrightarrow$-def,

$$z \in x \leftrightarrow z \in y$$

Add in $\forall$:

$$\forall z(z \in x \leftrightarrow z \in y)$$

Ax of Extensionality allows us to conclude

$x = y$. ■

def of $\forall$: $\forall x\, \varphi(x)$ means $\neg\exists\neg\varphi(x)$

Thm: $\forall y\,(\emptyset \subseteq y)$

Proof: (Proof by contradiction)

Assume $\neg\forall y\,(\emptyset \subseteq y)$

By def of $\forall$,

$\neg\neg\exists y\,\neg(\emptyset \subseteq y)$

By ~~def of $\exists$~~ dbl neg,

$\exists y\ \neg(\emptyset \subseteq y)$

Remove $\exists$:

$\neg(\emptyset \subseteq y_0)$

Def of $\subseteq$:

$\neg(\forall z(z\in\emptyset \to z\in y_0))$

Def of $\forall$:

$\neg\neg\exists z\ \neg(z\in\emptyset \to z\in y_0)$

Dbl neg:

$\exists z\ \neg(z\in\emptyset \to z\in y_0)$

Def of $\to$:

$\exists z\ \neg(\neg(z\in\emptyset) \lor (z\in y_0))$

De Morgan + dbl neg:

$\exists z\ z\in\emptyset \land \neg(z\in y_0)$

~~Empty set thm: $\forall x\ \neg(x\in\emptyset)$~~

~~Remove $\forall$ repl w/ z~~

~~$\neg(z\in\emptyset)$~~

Remove $\exists$:

$z_0\in\emptyset \land \neg(z_0\in y_0)$

$\land$-elim:

$z_0\in\emptyset$

Empty Set Thm:

$\forall y\,(\neg(y\in\emptyset))$

Remove $\forall$ – repl. y w/ z_0:

$\neg(z_0\in\emptyset)$

Thus we have: $z_0\in\emptyset \land \neg(z_0\in\emptyset)$ $\neg\forall y$

$\therefore$ Contradiction, hence our assumption is false — thus $\forall y\,(\emptyset\subseteq y)$

Prove $(\exists z)(\emptyset \in z)$

Pf: Let $\varphi(x)$ denote "$x = \emptyset$". Then Axiom 2 says

~~Prove~~ $\exists z \forall x (x \in z \leftrightarrow \varphi(x))$

Remove $\exists$:

$$\forall x (x \in z_0 \leftrightarrow \varphi(x))$$

$x = \emptyset$

Remove $\forall$ (repl w/ $\emptyset$)

$$\emptyset \in z_0 \leftrightarrow \emptyset = \emptyset$$

~~But~~ By $\leftrightarrow$-def,

$$\underbrace{\emptyset = \emptyset}_{\text{Tautology... so true!}} \rightarrow \emptyset \in z_0$$

So,

$$\emptyset \in z_0$$

Add in $\exists$:

$$\exists z (\emptyset \in z)$$